Management of Invasive Insect Species using Optimal Control Theory

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Abstract

We discuss the use of optimal control theory to determine the most cost-effective management strategies for insect pests. We use a stage-structured linear population projection model where the modeled control action increases the mortality in one of the stage-classes. We illustrate the method by using a published model for the root weevil *Diaprepes abbreviatus*, an invasive insect species having a substantial negative impact on citrus trees in regions such as Florida and California in the United States. Here control corresponds to the application of inundative biological control agents (entomopathogenic nematodes as biopesticides) which increases the mortality of the larval stage. Our approach determines levels and timing of control to minimize the economic loss caused by *D. abbreviatus*. We use two numerical methods to approximate the optimal control, and compare their effectiveness.

Keywords: population management, optimal control, invasive species, discrete-time system, *Diaprepes abbreviatus*

1. Introduction

Insect pests cause considerable economic loss to agriculture worldwide. This economic loss consists of reduced harvest and the cost of applying pesticide or biological control. The costs of pest control has been steadily increasing. For instance, from 2008-2012 the insecticide expenditure for producers increased worldwide from 12.5 to 16 billion US Dollars, and in the USA from 22 to 25 million US Dollars [1]. Further, pesticide usage incurs indirect environmental and economic costs associated with the recommended application of pesticides. These indirect costs include pesticide poisonings and illnesses of humans, domestic animals, and negative impact on beneficial animals [2]. In most cases the extensive use of pesticides also leads to the evolution of pesticide resistance which often necessitates an increase in insecticide application. As a response to these challenges some producers turn to using biological control methods including the release of natural

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enemies of the pest insects such as predators, parasitoids or pathogens [3, 4, 5]. Establishing a sufficiently high population of natural enemies can be challenging because of negative effects of disturbances caused by agronomic interventions such as tilling and the general low quality and diversity of agricultural landscapes. However, natural enemies can still be effective if they temporarily reach high numbers via augmentations. In this case, they function more like a pesticide that persists in the environment for a limited amount of time without any of the negative effects of chemical pesticides.

In this paper we present a mathematical framework for designing a management strategy that maximizes the profit for producers by using pest control strategies most effectively. We apply our framework to the management of the economically important citrus fruit insect pest, *Diaprepes abbreviates*, commonly referred to as Diaprepes root weevils, DRW [6]. DRW is a long-lived invasive insect species established in Florida and spreading into California. This insect species has four distinct life history stages (eggs, larva, pupae, and adults) of which the larvae are the by far most damaging stage. One promising strategy for targeting the larva stage is the application of commercially available entomopathogenic nematodes for inundative biological control (i.e., as biopesticides) [7, 8]. Entomopathogenic nematodes are parasites of insects that kill the infected host usually within 24 to 48 hours. They can be applied with most horticultural equipment including pressurized sprayers, mist blowers, and electrostatic sprayers, and are used as biopesticide for a range of different pest insects [9]. While there has been research on DRW [6]–[13], to the authors' knowledge this is the first use of optimal control theory to consider management options for DRW.

Specifically, we explore the use of optimal control theory for pest management, continuing a line of inquiry into optimal control in biological applications. Optimal control theory broadly refers to the area of mathematics and engineering where a control action is determined to achieve some desired dynamic behavior *and* minimize a prescribed cost functional. In the context of pest management, the desired dynamic behavior is a reduction in pest abundance whilst the cost functional models the combined cost of the loss of crop to pest and application of control. The reader is referred to [14, 15] for more background on optimal control theory, and to [16] for optimal control and its biological applications. The papers [17, 18, 19] outline general models for invasive species, specifically considering the invasive grass species, *Spartina alterniflora*. The paper [17] utilizes linear programming, while [18] implements linear-quadratic control with a spatially-explicit model using dynamic programming. The paper [19] incorporates endangered species into the model along with the invasive species. The papers [20, 21, 22] each model invasive species and use discrete optimal control theory to analyze the systems. In [20], part of the population has a nonlinear growth function, and they compare the impact of different choices of growth functions on the optimal control. The paper [21] formulates a discrete time optimal control problem for Gypsy Moths adding a control linearly into the system, where the control they are adding is also already naturally in the environment. Lastly, [22] considers discrete optimal control problems for invasive plant species, including budgetary constraints.

We propose a linear, stage-structured population model in discrete time representing a generalized life cycle of an insect species [23] and consider a control action that reduces survival of a single stage-class. For our model, the control is incorporated into the state equations through a nonlinear function. This type of control could apply to biopesticides like entomopathogenic nematodes or chemical insecticide applications using products affecting a single stage-class. For instance, for species where the larval stage lives in the soil, toxins may be sprayed on the surface and kill newly hatched larvae while burying into the soil [7]. We consider two different types of control. First, we assume that each control action affects the insects for a single time-step only. Second, we consider control which persists, but decays exponentially over time post application. We define a cost functional to describe loss in harvest income from the pest plus the cost of applying the control. We say that a control strategy is optimal if it minimizes the cost functional subject to the dynamics of the controlled pest population. The optimal amount and timing of control application should balance the cost of the control with the cost of allowing the pest to reduce the harvest. The resulting optimal control is likely to be time dependent. For instance, it might be best to start applying control when pest density is low and keep applying control over the entire season (frequent control at low intensity). Alternatively, it might be better to carry out a small number of control actions with high intensity and allow pest densities to build up between control actions. Optimal control techniques can help decide between competing control schemes, even if the optimal control is difficult to apply.

We compare two different numerical search algorithms using MATLAB to solve the optimal control problem, since finding the optimal controls analytically is typically intractable. The Forward-Backward Sweep is a very efficient method for approximating optimal controls and relies on the Pontryagin Maximum Principle [24], via a dynamical system called the adjoint system [16, Chp. 4, 23]. However, we will see that this method breaks down if the control decays slowly over time. In our second method we implement a standard MATLAB function called MultiStart, which does not rely on the adjoint system, but is much more computationally intensive. With enough computational runs MultiStart produces a reasonable solution, but is less accurate than Forward-Backward Sweep.

This paper is organized as follows. In Section 2, we formulate the optimal control problem for a model for a general insect species with control, specify a cost function, and establish the existence of solutions to the optimal control problems. We illustrate our method with a case study on *Diaprepes abbreviatus*, DRW [6]. In Section 3, we give numerical simulations of our DRW case study. We discuss our results and their significance in Section 4.

2. Methods

2.1. Model Formulation

We start this section with a matrix model for the dynamics of an invasive insect pest. We model the pest in four distinct stage-classes, denoted by $P_1(t)$, $P_2(t)$, $P_3(t)$ and $P_4(t)$, at time-step t. Here the time variable $t \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ denotes how many time-steps have passed. We denote the four-dimensional vector with these stages by

$$\mathbf{P}(t) := \begin{bmatrix} P_1(t) & P_2(t) & P_3(t) & P_4(t) \end{bmatrix}^\top, \quad t \in \mathbb{Z}_+ ,$$

where the superscript T denotes vector transposition. All of the models we consider are local in the sense that there is no explicit spatial dependence or variation, only temporal. For the applications we have in mind, the stage-classes shall denote the abundance of the four distinct developmental stages: eggs, larvae, pupae and adults. The associated population projection matrix is the 4×4 matrix:

$$A = \begin{bmatrix} \gamma_1 & 0 & 0 & \theta_1 \\ \gamma_2 & \zeta_1 & 0 & 0 \\ 0 & \zeta_2 & \nu_1 & 0 \\ 0 & 0 & \nu_2 & \theta_2 \end{bmatrix},$$

where θ_j, γ_j, ν_j, and ζ_j are nonnegative parameters, for $j \in \{1, 2\}$. The descriptions for the matrix A parameters are in Table (2.1). We assume that a scalar control, denoted by $N(t)$, is applied at each time-step. We consider the situation where the control is applied to only one stage-class; here we choose the second stage-class P_2, but it can be applied to any stage. To capture saturation effects reflecting diminishing returns for large control efforts, we assume that the control efficacy is given by $f(N(t))$, for a given function $f : \mathbb{R}_+ \to \mathbb{R}_+$. We typically assume that f is nonincreasing. The dynamics of the pest with control are described by:

$$\mathbf{P}(t+1) = \mathcal{A}(N(t))\mathbf{P}(t) \quad \mathbf{P}(0) = \mathbf{P}^0 \quad t \in \mathbb{Z}_+ , \tag{2.1}$$

where

$$\mathcal{A}(N(t)) = \begin{bmatrix} \gamma_1 & 0 \cdot f(N(t)) & 0 & \theta_1 \\ \gamma_2 & \zeta_1 \cdot f(N(t)) & 0 & 0 \\ 0 & \zeta_2 \cdot f(N(t)) & \nu_1 & 0 \\ 0 & 0 \cdot f(N(t)) & \nu_2 & \theta_2 \end{bmatrix} \quad \text{and} \quad \mathbf{P}^0 = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}$$

denote the nonlinear projection matrix and initial population, respectively. Note that the model counts the transitions from one population census to the next, where we count immediately after birth. Hence, the model assumes that the control does not impact the transition from P_1 to P_2. Figure 2.1 depicts the model dynamics in equation (2.1).

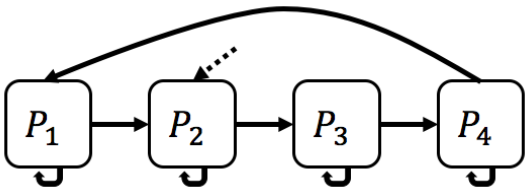


Figure 2.1: Life-cycle diagram modeled by equations (2.1). The solid lines represent either survival or fecundity rates. The dotted line represents the control N acting on stage P_2.

We comment that the model (2.1), and the approach we take, generalizes to higher (but finite) numbers of stage-classes. Specifically, for $n \in \mathbb{Z}_+$, $\mathbf{P}$ can be replaced by

$$\begin{bmatrix} P_1(t) & P_2(t) & \dots & P_n(t) \end{bmatrix}^\top, \quad t \in \mathbb{Z}_+,$$

and A can be replaced by the $n \times n$ matrix

$$\begin{bmatrix} \gamma_1^1 & 0 & \dots & 0 & \gamma_1^n \\ \gamma_2^1 & \gamma_1^2 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \gamma_1^{n-2} & \gamma_1^{n-1} & 0 \\ 0 & \dots & 0 & \gamma_2^{n-1} & \gamma_2^n \end{bmatrix},$$

where γ_j^i are nonnegative parameters for $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2\}$. Similarly, the controlled dynamics $\mathcal{A}(N(t))$ can be extended to the to n stage-classes in the obvious way.

2.2. Cost Functional

In order to formulate an optimal control problem, we construct a cost functional with two contributing terms for each time step: the loss in profit owing to insect related crop damage, and the cost of purchasing and applying control. We use the quadratic term $\beta_1 P_2(t)^2$ to model the cost of pest damage during time-step t, which is a commonly used term in optimal control problems (see [16, Chp. 2]), where β_1 weights the relative importance of the pest damage. Such a quadratic term heavily penalizes catastrophic loss, for instance in a situation where farmers grow valuable perennial plants such as fruit trees, and plant death caused by high pest density would add significant extra costs such as the cost of planting new plants and lost income due to a delay in harvest of several years.

We use the linear term $\beta_2 N(t)$ to model the cost of purchasing and applying the control agent $N(t)$ at time-step t, where β_2 is the price of a single control unit. In the absence of other information about this cost, such a linear term seems reasonable. The total cost in the t^{th} time-step is then given by $\beta_1 P_2(t)^2 + \beta_2 N(t)$.

Finally, we fix $T \in \mathbb{Z}_+$ which denotes the time window over which the optimal control problems are solved. Given a sequence of T control actions $N(0), \ldots, N(T-1)$ and resulting states $\mathbf{P}(t)$ given by (2.1), we define the cost functional J by

$$J(N) = \sum_{t=0}^{T-1} \left[\beta_1 P_2(t)^2 + \beta_2 N(t)\right] . \tag{2.2}$$

We are now in position to formulate two optimal control problems.

2.3. Optimal Control Problem Model 1: control lasts one time step

The first optimal control problem we consider in this paper, which we call Model 1, is to minimize $J(N)$ given by (2.2), subject to (2.1) over all nonnegative control functions:

$$N \in \mathbf{N} = \big\{ N(t) \mid 0 \leq N(t) \text{ for } t = 0, 1, ..., T-1 \big\}. \tag{2.3}$$

2.4. Optimal Control Problem Model 2: control decays exponentially

In Model 1 the control $N(t_j)$ is assumed to be only active on $[t_j, t_{j+1})$. However, biological control agents may persist for some time in the environment, and will remain efficacious, especially when the time-step is short (for instance, a day or a week instead of a season). To account for this decay we augment N with its own dynamics: we let $N_{\mathrm{n}}(t)$ denote the new control action at time-step t, and N_{o} denote remains of control actions from previous time-steps.

The second optimal control problem, which we call Model 2, is: minimize J given by (2.2), now subject to the dynamics

$$\left.\begin{aligned} N_{\mathrm{o}}(t+1) &= N_{\mathrm{o}}(t)\mathrm{e}^{-\mu} + N_{\mathrm{n}}(t) & N_{\mathrm{o}}(0) &= 0 \\ \mathbf{P}(t+1) &= \mathcal{A}(N_{\mathrm{o}}(t)e^{-\mu} + N_n(t))\mathbf{P}(t) & \mathbf{P}(0) &= \mathbf{P}^0 \end{aligned}\right\} \quad t \in \mathbb{Z}_+ , \tag{2.4}$$

and the cost constraint (2.3) for N_{n}, namely

$$N_{\mathrm{n}} \in \mathbf{N} .$$

In Model 2 we assume that the biological control agents decay exponentially with rate μ, leading to the first dynamic equation in (2.4). Note that Model 2 resembles Model 1 for large μ (meaning that the control agents persist for a short time).

The existence of optimal controls for Models 1 and 2 is given in the following theorem. A proof of uniqueness of these solutions is more complicated, due to the nonlinearity in the control, and is not addressed in this paper. We will state and prove the theorem for Model 2, since the proof for Model 1 is identical.

Theorem 1. *For all initial conditions $\mathbf{P}^0 \in \mathbb{R}^4_+$, there is a control $\mathcal{N} \in \mathbf{N}$ which minimizes J subject to* (2.4)*.*

Proof. There exists an N_{max} such that $\beta_2 N_{max} > J(0)$. If $N(t) > N_{max}$ for any t, $J(N) > J(0)$ (where 0 denotes the function that is identically equal to zero). Hence the mimumum of J over $\mathbf{N}$ is the same as the minimum of J over

$$\mathbf{N}_{max} = \big\{N(t) \mid 0 \leq N(t) \leq N_{max}\big\}.$$

Let $\mathcal{T} : \mathbf{N}_{max} \subset \mathbb{R}^T \to \mathbb{R}^{4\times T}$ be defined by

$$(\mathcal{T}(N))_{ij} = P_i(j) \quad \text{for all } i \in \{1,2,3,4\} \text{ and } j \in \{0,1,\dots T-1\},$$

where $P_i(j)$ is the i-th component of $\mathbf{P}(j)$, the solution of (2.4) at time-step j. Then, from (2.4), it follows that $\mathcal{T}$ is continuous with any choice of norms for $\mathbf{N}_{max}$ and $\mathbb{R}^{4\times T}$. Hence J is continuous from $\mathbf{N}_{max}$ to $\mathbb{R}$. Since $\mathbf{N}_{max}$ is a compact subset of $\mathbb{R}^T$, J takes on a global minimum at some $N \in \mathbf{N}_{max} \subseteq \mathbf{N}$. □

2.5. Case Study: Diaprepes abbreviatus

As a motivating and potentially economically important example, we consider the insect pest species *D. abbreviates*, commonly called Diaprepes root weevils, (DRW). DRW originated in the Caribbean and was transported to the central and southern regions of Florida around 1964 [6]. The introduction of DRW was not intentional, and in the past 50 years DRW has become a troublesome invasive species, spreading throughout Florida and eventually to California in 2005 [6, 13]. DRW infests citrus groves along with other plants [12]. Adult DRW feed and lay eggs on plants. Upon hatching, neonate larvae drop off the leaves and bury into the soil, where they feed on plant roots for several months until they pupate. The damage caused by the feeding activity of the larvae can be severe, causing tree decline and death. DRW dynamics have been described in a matrix model in Miller and Tenhumberg [10]. In the past, halogenated hydrocarbon insecticides were applied to the soil to control DRW populations. This class of insecticide was deregistered in 1984 [25], and since then there has been a lack in pesticides that adequately manage the DRW larva [26]. An alternative to pesticides is the usage of inundative biological control agents such as entomopathogenic nematodes attacking DRW larvae [7, 8]. Nematodes are utilized like pesticides, requiring regular applications [26]. We explore two different scenarios. The first scenario assumes that nematodes cannot persist and their life expectancy is only approximately one week [27], see Model 1 in Section 2.3. The second scenario assumes nematode populations decline exponentially, see Model 2 in Section 2.4. We determine management plans for both scenarios specifying timing and amount of nematodes applications, while also considering the cost of applying nematodes and the cost of DRW damage to the farmer.

2.5.1. DRW Parameter Values

To use the framework of Models 1 and 2, we reduced the 6×6 population projection matrix from [10] for DRW to a 4×4 matrix using Hooley's algorithm [28]. The four stage-classes of the matrix denote eggs (P_1),

larva (P_2), pupa (P_3), and adults (P_4), and the time-steps denote weeks, with parameter values in Table (2.1) scaled accordingly. The resulting projection matrix is given by

$$A = \begin{bmatrix} \gamma_1 & 0 & 0 & \theta_1 \\ \gamma_2 & \zeta_1 & 0 & 0 \\ 0 & \zeta_2 & \nu_1 & 0 \\ 0 & 0 & \nu_2 & \theta_2 \end{bmatrix} = \begin{bmatrix} 0.305 & 0 & 0 & 80.447 \\ 0.530 & 0.899 & 0 & 0 \\ 0 & 0.020 & 0.778 & 0 \\ 0 & 0 & 0.202 & 0.980 \end{bmatrix}.$$

Hooley's algorithm guarantees that the dominant eigenvalue, 1.423, of the reduced matrix, A, is the same as that of the original matrix [10]. Since the dominant eigenvalue is greater than 1, the model predicts growth of the uncontrolled population. Interpretations of the above parameter values, along with all other parameter values arising in the optimal control problems, are presented in Table (2.1).

The choice of optimal control depends on the distribution of the initial population among stages-classes, and its size, captured by some norm of the initial population $\mathbf{P}^0$. We choose the stable-stage distribution (SSD) as an initial distribution to minimize the effect of transient dynamics on our control. The SSD is a multiple of the normalized eigenvector associated with the dominant eigenvalue of a population projection matrix. We then scaled the SSD by different constants to explore the effect of different initial DRW population sizes. Starting with the SSD is relevant to a situation where the pest species has persisted in an orchard for several years without adding any control because DRW escaped detection. Founder populations are typically small and for several generations they may remain sufficiently small to cause no notable damage even if the population grows at a constant rate. Hence, it is plausible that populations are at the SSD before farmers apply control measures, although the methods we propose are applicable for any $\mathbf{P}^0$.

2.5.2. The saturation function f

As mentioned above, we will model the application of entomopathogenic nematodes, where the amount of control at time t is denoted by $N(t)$ and $N_{\mathrm{n}}(t)$ in Models 1 and 2, respectively. The control is applied only to the second stage-class P_2 and relies on control agent-host interactions. We assume that the control agent-host encounters follow a Poisson distribution, so we use the saturation function $f(N) = e^{-\alpha N}$, where α denotes the control search/application efficiency, which accounts for how likely a control agent is to encounter a pest larva. We could not find an estimate of α in the literature. Therefore, we made the assumption that the recommended number of nematodes per hectare, $N(\cdot) = N_S = 22 \times 10^8$ per hectare ([7]), results in population stasis, and iteratively searched for α values that produced a slightly increasing population density. This would likely be an underestimation of α (conservative estimate for the effacing of the nematodes), so we explore the control strategies using a range of α values.

When $\alpha = 1.655 \times 10^{-8}$ and $N(\cdot) = N_S$, the population projection matrix has a dominant eigenvalue of 1.000005, meaning the model (2.1) predicts that the DRW population growth very slowly. This choice of α

predicts that the recommended dose according to the manufacture specifications is not sufficient to produce population decline. Hence, we would expect that our model predicts higher than recommended nematode applications. By using an α value that is slightly too low our management recommendation has a "safety net" built in, since the control might still work if nematode efficiency is low owing to poor environmental conditions. Owing to the uncertainty in the choice of α we will vary the parameter to evaluate its effect on the total cost of the system in numerical simulations.

2.5.3. Parameter values for the cost function

In the literature we could not find an estimate for β_1, which specifies how much damage a single DRW larvae causes in terms of loss in harvest. To ensure that our choice for β_1 is sensible we compare the first part of our cost functional with natural densities of DRW larvae and the average income of farmers from selling citrus fruits [7, 11]. One study reported DRW larvae densities in Florida of 376.25 per m^2 in non-irrigated orchards and 945.51 per m^2 in irrigated orchards [11], but the study did not report on the associated harvest. In Figure 2.2 the vertical lines are these numbers scaled for DRW larvae per hectare. Further, in 1999 farmers made on average US$6000 per hectare for citrus [7]. According to the Federal Reserve Bank of Minneapolis, the annual average consumer price index was 166.6 in 1999 and is 244.3 in 2017 [29], indicating an inflation of 46.6% [30]. Hence, in 2017 the expected weekly income of a citrus farmer in Florida is US$170 (=(US$6000 $\times$ 1.466)/52; horizontal line in Figure 2.2). Our choice of β_1 implies that at a DRW larvae population of 4.62381×10^6 produces zero income for the farmer. In this paper we also varied β_1 up and down by 15% (thick and thin solid lines in Figure 2.2). All three values for β_1 are conservative estimates of the impact of DRW larvae on citrus harvest. Applying our method to a real world management situation would require improved estimates for β_1.

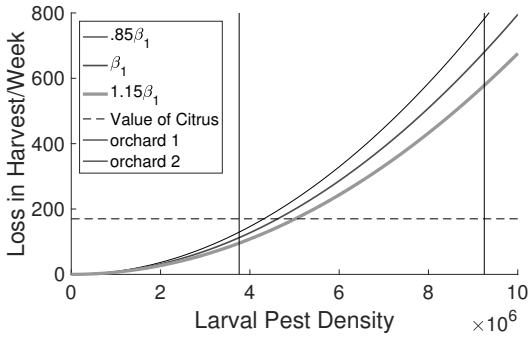


Figure 2.2: Plot for DRW cost portion. The above illustrates the β_1 value in the cost function, showing ranging β_1 up and down by 15%. The vertical lines are DRW larval pest density in two orchards as minimum and maximum values. The horizontal line show the value of citrus a farmer can expect in harvest per week.

At the time of this writing, a batch of nematodes that is recommended by industry to control DRW can

be purchased for \$62 per hectare [7]. Additionally, it is recommended to use 22 nematodes per cm^2 [7]. To find the value of β_2 we combined these values and scaled the value for a hectare and a single nematode. Note that while we have configured the cost to be per one nematode, typically nematodes are purchased in bulk. Therefore we will again vary the value of β_2 in simulations for the optimal control problem. All parameter values are shown in Table 2.1.

	Notation	Description	DRW Value	Source
Pest Matrix	γ_1	Egg survival	0.305	[10]
	γ_2	Transition rate from egg to larva	0.530	[10]
	θ_1	Fecundity rate of female adults	80.447	[10]
	θ_2	Adult survival	0.980	[10]
	ζ_1	Larva survival	0.899	[10]
	ζ_2	Transition rate from larva to pupa	0.020	[10]
	ν_1	Pupa survival	0.778	[10]
	ν_2	Transition rate from pupa to adult	0.202	[10]
Initial Pest Vector	ϕ_1	Initial Proportion of eggs	0.486	[10]
	ϕ_2	Initial Proportion of larva	0.492	[10]
	ϕ_3	Initial Proportion of pupa	0.015	[10]
	ϕ_4	Initial Proportion of adults	0.007	[10]
Control	α	encounter rate of nematodes with DRW larva	1.655×10^{-8} $[1.4068, 1.903] \times 10^{-8}$	[7]
Cost Function	β_1	Costs due to tree damage in US dollars per hectare per week per DRW larva^2	7.952×10^{-12} $[6.760, 9.145] \times 10^{-12}$	[7, 11]
	β_2	Cost of control in US dollars per hectare per week per nematode	2.818×10^{-8} $[2.360, 3.241] \times 10^{-8}$	[7, 11, 30]
Model 2	μ	nematode mortality rate	$\ln(2)$ $[1/26, 100]$	*

Table 2.1: Parameter values for DRW. The * denotes that we do not have a source for the estimate, and will vary it in simulations.

3. Results

In Theorem 1, we established that optimal controls exist for Models 1 and 2. To identify these optimal controls, we use an the extension of Pontryagin Maximum Principle [31] - for the original principle, see [24]. Suppose that $\mathcal{N}$ is an optimal control for Model 1. Let $\mathcal{P}_2$ be the second component of the solution $\mathbf{P}$ of (2.1) when subject to the optimal control $\mathcal{N}$. The following is referred to as the *adjoint* system for (2.1):

$$\left.\begin{aligned}
\lambda_1(t) &= \lambda_1(t+1)\gamma_1 + \lambda_2(t+1)\gamma_2 \\
\lambda_2(t) &= 2\beta_1\mathcal{P}_2(t) + \lambda_2(t+1)\zeta_1 e^{-\alpha\mathcal{N}(t)} \\
&\quad + \lambda_3(t+1)\zeta_2 e^{-\alpha\mathcal{N}(t)} \\
\lambda_3(t) &= \lambda_3(t+1)\nu_1 + \lambda_4(t+1)\nu_2 \\
\lambda_4(t) &= \lambda_1(t+1)\theta_1 + \lambda_4(t+1)\theta_2 \\
\lambda_1(T) &= 0,\ \lambda_2(T) = 0,\ \lambda_3(T) = 0,\ \lambda_4(T) = 0
\end{aligned}\right\}\ t \in \{0, 1, \ldots, T-1\}\,. \tag{3.1}$$

The adjoint variables λ_1, λ_2, λ_3, and λ_4 perform a function similar to that of Lagrange multipliers. A necessary condition for $\mathcal{N}$ to be a solution of the optimal control problem for Model 1 (see [32, Chp. 2]) is that $N(t) = \mathcal{N}(t)$, $\mathbf{P}(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, and $\lambda_4(t)$ solve (2.1), (3.1) and

$$\mathcal{N}(t) = \begin{cases} 0 & \text{if } \frac{\beta_2}{\alpha} > \xi(t) \\ \frac{1}{\alpha}\ln[\frac{\alpha}{\beta_2}\xi(t)] & \text{if } \frac{\beta_2}{\alpha} \leq \xi(t) \end{cases}, \tag{3.2}$$

where $\xi(t) = \zeta_1\lambda_2(t+1)\mathcal{P}_2(t) + \zeta_2\lambda_3(t+1)\mathcal{P}_2(t)$. If we can solve (2.1), (3.1), and (3.2), then the solution will be an optimal control [16, Chps. 1, 23].

For Model 2, we again use the extension of Pontryagin Maximum Principle. Suppose that $\mathcal{N}_n$ is an optimal control for Model 2. Let $\mathcal{P}_2$ be the second component of the solution $\mathbf{P}$ and $\mathcal{N}_o$ the solution $\mathbf{N}_o$ of (2.4) when subject to the optimal control $\mathcal{N}_n$. The following is referred to as the *adjoint* system for (2.4):

$$\left.\begin{aligned}
\lambda_1(t) =& \lambda_1(t+1)\gamma_1 + \lambda_2(t+1)\gamma_2 \\
\lambda_2(t) =& 2\beta_1\mathcal{P}_2(t) + \lambda_2(t+1)\zeta_1 e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))} \\
&+\lambda_3(t+1)\zeta_2 e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))} \\
\lambda_3(t) =& \lambda_3(t+1)\nu_1 + \lambda_4(t+1)\nu_2 \\
\lambda_4(t) =& \lambda_1(t+1)\theta_1 + \lambda_4(t+1)\theta_2 \\
\lambda_o(t) =& -\alpha\zeta_1 e^{-\mu}\lambda_2(t+1)e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_2(t) \\
&-\alpha\zeta_2 e^{-\mu}\lambda_3(t+1)e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_2(t) + \lambda_o(t+1)e^{-\mu} \\
\lambda_1(T) =& 0,\ \lambda_2(T) = 0,\ \lambda_3(T) = 0,\ \lambda_4(T) = 0,\ \lambda_o(T) = 0.
\end{aligned}\right\}\ t \in \{0, 1, \ldots, T-1\} \tag{3.3}$$

A necessary condition for $\mathcal{N}_n$ to be a solution of the optimal control problem for Model 2 is that $N(t) = \mathcal{N}(t)$, $\mathbf{P}(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, and $\lambda_4(t)$ solve (2.4), (3.3) and

$$\mathcal{N}_n(t) = \begin{cases} 0 & \text{if } e^{\alpha \mathcal{N}_o(t) e^{-\mu}} > \xi_n(t) \\ \frac{1}{\alpha} \ln(\xi_n(t)) - \mathcal{N}_o(t) e^{-\mu} & \text{if } e^{\alpha \mathcal{N}_o(t) e^{-\mu}} \leq \xi_n(t) \end{cases} \tag{3.4}$$

with

$$\xi_n(t) = \frac{\alpha \mathcal{P}_2(t)[\lambda_2(t+1)\zeta_1 + \lambda_3(t+1)\zeta_2]}{\beta_2 + \lambda_o(t+1)}.$$

If we can solve (2.4), (3.3), and (3.4), then the solution will be an optimal control

As mentioned above, we have not established uniqueness. However, as we will see in the numerical results in Section 3.1, there is no indication that the optimal controls are non-unique.

3.1. Numerical Simulations for DRW

We cannot explicitly solve the systems [(2.1) (3.1) (3.2)] or [(2.4) (3.3) (3.4)] analytically, so we use numerical simulations with the DRW parameter values from Table 2.1. For the numerical simulations we used two different methods in MATLAB, Forward-Backward Sweep (FBS) and MultiStart, which will be described in Sections 3.1.1 and 3.1.4 respectively.

3.1.1. Forward-Backward Sweep (FBS)

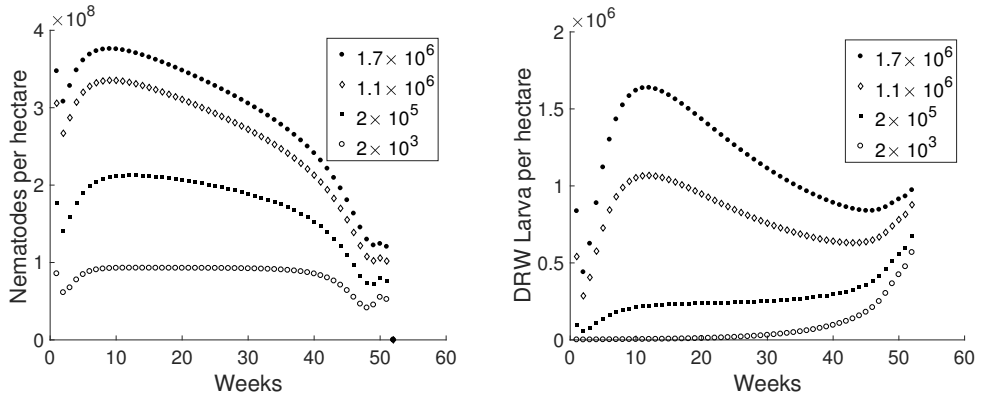
Forward-Backward Sweep (FBS) is an algorithm typically used to find an estimate for the solution to an optimal control problem [16, Chp. 4, 23]. The algorithm takes an estimate for the control and calculates the associated state system and adjoint system. Then, with the new state and adjoint systems, the algorithm calculates the control. Next, the algorithm checks if the difference between the new values and the previous values are smaller than a chosen acceptable error value. Once the difference between estimates is acceptable, the algorithm stops and outputs the last control values. We will use the following process:

1. Let $N = 0$ and use (2.1) or (2.4) to calculate $\mathbf{P}$ (and N_o) from initial conditions.
2. Now calculate $\lambda_1, \lambda_2, \lambda_3, \lambda_4, (\lambda_o)$ using (3.1) or (3.3).
3. Using the calculations in 1 and 2, find N.
4. Check if the differences between the newly calculated $\mathbf{P}$, (N_o), λ_s, and N, for $s = 1, 2, 3, 4, (o)$, are within an acceptable error. If so, stop, since you have an acceptable estimate for the optimal control. If not, use the N in Step 3 and repeat the process.

3.1.2. Model 1

We varied the initial population of DRW in the FBS method for Model 1, to see how the optimal application of nematodes, and corresponding DRW larvae population, depend on this initial population. See

Figure 3.1.2, as expected the larger the initial pest density the larger the number of nematodes in the optimal control. But interestingly, the resulting DRW larval density obtained with the optimal control also increases. In other words, if initial pest density is high it is not optimal to get pest density as low as when initial pest density is low. This is different from the automatic spray schedule used by farmers. With the optimal control problem we are considering how to minimize the total combined cost, possibly applying control for each time step.



(a) (b)

Figure 3.1: (a) We used the FBS on Model 1 to calculate the number of nematodes to apply for various initial DRW populations per hectare. (b) Next, we have the corresponding DRW larva populations for the nematode application in (a) for various initial DRW populations per hectare.

In section 2.5.2 we gave a conservative value for search efficiency, α. Since this parameter is highly uncertain we ran numerical simulations with a range of different α values. We also explored the effect of varying the parameter values for β_1 and β_2 from section 2.5.3. We varied the three parameters assuming the initial population is 1.7×10^6 DRW per hectare. The results are displayed in Figure 3.2. We fitted the simulation results to linear functions; the least square regression lines explain >99% of the variation. The results show that the total costs increase with increasing β_1 and β_2, and the total costs decrease with increasing α.

3.1.3. Model 2

While the FBS is a useful algorithm for many problems, there are circumstances that cause the algorithm to not converge [33]. In our Model 2 optimal control problem, for many choices of μ FBS cannot be used to find the optimal number of nematodes to apply. In fact, in many simulations the algorithm gives a zero control, which is not close to optimal. This can be explained as follows. Looking at (3.4) for nematodes, N_{n}:

$$N_{\mathrm{n}}(t) = \begin{cases} 0 & \text{if } e^{\alpha N_{\mathrm{o}}(t)e^{-\mu}} > \frac{\alpha P_2(t)[\lambda_2(t+1)\zeta_1+\lambda_3(t+1)\zeta_2]}{\beta_2+\lambda_{\mathrm{o}}(t+1)} \\ \frac{1}{\alpha}\ln[\frac{\alpha P_2(t)[\lambda_2(t+1)\zeta_1+\lambda_3(t+1)\zeta_2]}{\beta_2+\lambda_{\mathrm{o}}(t+1)}] - N_{\mathrm{o}}(t)e^{-\mu} & \text{if } e^{\alpha N_{\mathrm{o}}(t)e^{-\mu}} \leq \frac{\alpha P_2(t)[\lambda_2(t+1)\zeta_1+\lambda_3(t+1)\zeta_2]}{\beta_2+\lambda_{\mathrm{o}}(t+1)} \end{cases}. \tag{3.5}$$

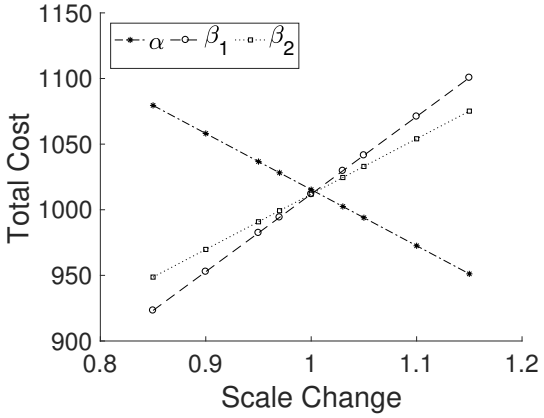


Figure 3.2: We used FBS on Model 1 with initial DRW population of 1.7×10^6 per hectare. To see the effect of α, β_1, and β_2 on total cost, we varied the values by scale amounts, and plotted these varied values versus the resulting total cost.

In order for (3.5) to predict a non-zero optimal control, that is, $N_{\rm n}(t) > 0$, it is necessary that that

$$e^{\alpha N_{\rm o}(t)e^{-\mu}} \leq \frac{\alpha P_2(t)[\lambda_2(t+1)\zeta_1 + \lambda_3(t+1)\zeta_2]}{\beta_2 + \lambda_{\rm o}(t+1)}\,. \tag{3.6}$$

The larger μ is, the "easier" it is to satisfy (3.6), as the left hand side decreases with increasing μ. We know that $1 \leq e^{\alpha N_{\rm o}(t)e^{-\mu}}$ since $N_{\rm o}(t)e^{-\mu} \geq 0$, so $\frac{\alpha P_2(t)[\lambda_2(t+1)\zeta_1+\lambda_3(t+1)\zeta_2]}{\beta_2+\lambda_{\rm o}(t+1)}$ will need to be positive. We have that $\alpha P_2(t)[\lambda_2(t+1)\zeta_1 + \lambda_3(t+1)\zeta_2] > 0$. We also need to consider $\beta_2 + \lambda_{\rm o}(t+1)$. Recall that

$$\begin{aligned}\lambda_{\rm o}(t) = &-\alpha\zeta_1 e^{-\mu}\lambda_2(t+1)e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_2(t)\\ &-\alpha\zeta_2 e^{-\mu}\lambda_3(t+1)e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_2(t) + \lambda_{\rm o}(t+1)e^{-\mu}.\end{aligned}$$

Since $\lambda_{\rm o}(T) = 0$, $\lambda_{\rm o}(t) < 0$ for all t, which means for $\beta_2+\lambda_{\rm o}(t+1)$ to be positive we need that $\beta_2 > -\lambda_{\rm o}(t+1)$, as β_2 is a positive constant. In our case study, $\beta_2 = 2.8182 \times 10^{-8}$. When $0 < \mu < 1$ it is not likely that $\beta_2 > \lambda_{\rm o}(t+1)$, so the result of FBS is always $N_{\rm n}(t) = 0$, or does not converge.

Figure 3.3 shows the effect on the FBS solution of varying μ for an initial DRW population of 1.1×10^6 per hectare. Note that for small values of μ the FBS produces a solution that is all zeros, or starts with zeros. When μ is small, more of the nematodes survive a week, but the FBS solution is instructing us to never apply nematodes, or wait until much later into the application period. Meanwhile, for larger values the optimal control is similar to that for Model 1. From this we see that FBS is not a reliable method for solving the optimal control problem when μ is small. We will see that we can often obtain a lower total cost with an estimate of the control that is based on a different numerical method.

3.1.4. MultiStart

The MultiStart (MS) algorithm is a standard MATLAB function which implements the `fmincon` function from MATLAB. The `fmincon` function searches for a local minimum near a user defined starting value. For

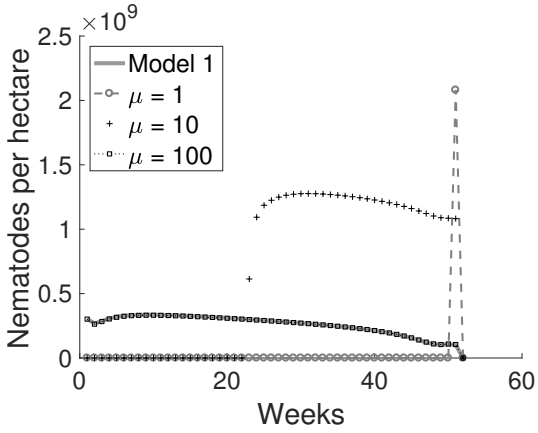
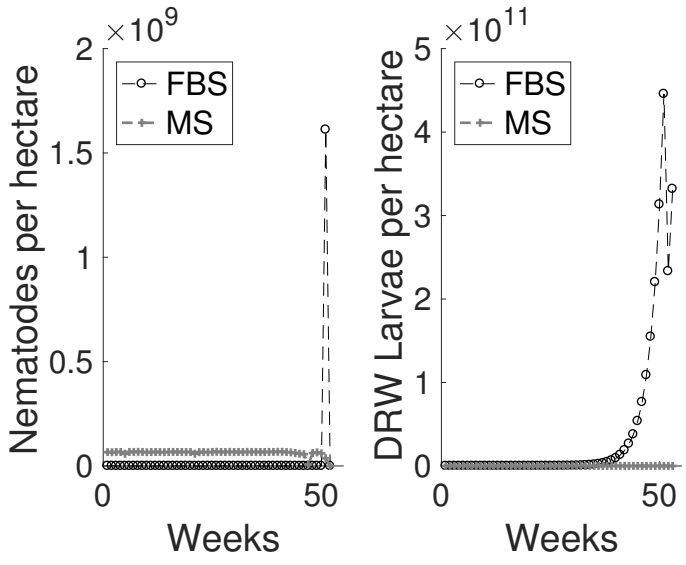


Figure 3.3: We compare Model 1 and 2 with initial DRW population of 1.1×10^6 per hectare. For the Model 2 simulations we used $\mu = 1, 10, 100$. As we increase μ, Model 2 is close to Model 1, but for small values of μ we can observe unreliability of the FBS simulation results.

the algorithm we input the dynamical system, objective functional, and parameter values. Unlike FBS, this algorithm does not use the adjoint equations. To find the global minimum MultiStart uses many randomly generated points as starting control values and picks the strategy associated with the lowest local minimum. The results we display use 500,000 randomly chosen points, we used MATLAB 2013b.

In Figure 3.4 we compare MultiStart to FBS for $\mu = \ln(2)$ with Model 2, a value which caused the FBS algorithm to no longer converge. We can see the FBS algorithm nematode application is zero except for one large value and the resulting total cost in dollars is 3.55×10^{12} with a general exponential growth of DRW larvae. Meanwhile, MultiStart produces a more constant application management plan and has a total cost of 1.01×10^2 with far fewer DRW larvae surviving.

To evaluate how accurately MultiStart can solve the optimal control problem, we compared MultiStart to FBS for Model 1, for two different initial DRW densities, see Figure 3.5. For an initial DRW population of 2×10^5 per hectare the total cost using FBS is 287.35 and 300.47 with MultiStart. When we use an initial DRW population of 1.7×10^6 per hectare the FBS total cost is 1012.16 and the MultiStart total cost is 1047.23. As we increased the number of runs with MultiStart the difference between the methods' total costs decreased. The comparison between MutiStart and FBS shows that the solution using MultiStart is satisfactory for many purposes. We can also note that the MultiStart results have some fluctuation, which is due to the algorithm searching a large number of values for a solution. As we increase the number of runs, the MultiStart results have fewer oscillations, but the run time increases. In the remainder of the paper we apply the MutiStart algorithm for Model 2 and for brevity we refer to the result provided by MultiStart as "optimal" nematode application strategy even though the result is not truly optimal.



(a) (b)

Figure 3.4: We compare FBS and MS for Model 2 with initial DRW population 2×10^5 per hectare and $\mu = \ln(2)$. In (a) we plot number of nematodes to apply and in (b) the corresponding DRW larvae populations.

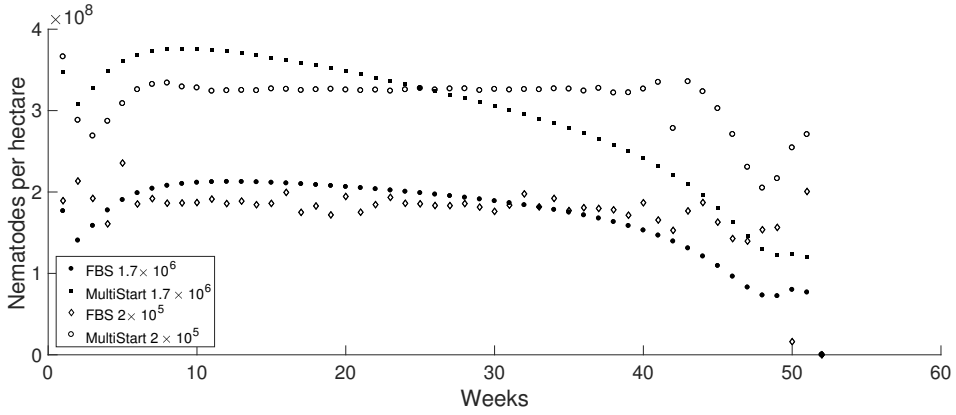


Figure 3.5: We compare FBS and MultiStart for Model 1 with initial DRW population of 2×10^5 per hectare and and 1.7×10^6 per hectare. We plot number of nematodes to apply for post initial DRW population per hectare.

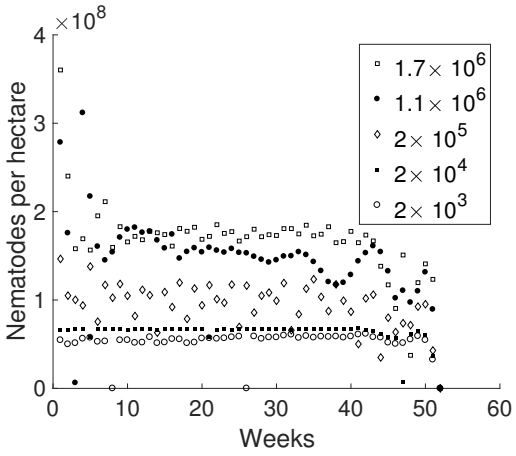
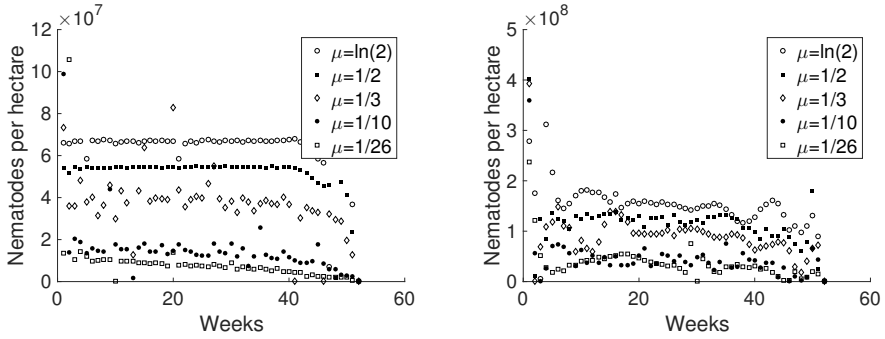


Figure 3.6: We use MS for Model 2 with $\mu = \ln(2)$ while we vary the initial DRW populations per hectare.

We explore the effect of changing the initial DRW population for $\mu = \ln(2)$ (see Figure 3.6). Note from about weeks 6 to 42 the simulations are seemingly converging to a constant amount of nematodes to apply each week. With the larger initial DRW population there are more erratic variations in the solution, which reduce with more runs in MultiStart. In Figure 3.7 we plot the optimal number of nematodes to apply when varying values for μ with initial DRW populations of 2×10^5 and 1.1×10^6 per hectare. Note that, as expected, the longer a released nematode population persists the fewer nematodes are needed for controlling DRW. Also, as in Model 1 we need to apply more nematodes if the initial DRW infestation is high.



(a) (b)

Figure 3.7: We use MS on Model 2, for (a) an initial DRW population of 2×10^4 per hectare and varying μ. Meanwhile in (b) we use an initial populations of 1.1×10^6 per hectare and vary μ.

We noted the apparent convergence of nematode application to a constant in Figure 3.6. Thus it seems plausible that a constant control application will be effective. Constant application amounts would be appealing for farming applications. We choose a constant control which best fits in a least squares sense our estimate for the optimal control obtained using MultiStart. Additionally, we ran the MultiStart algorithm

for applying a constant amount of control $\bar{N}$ at each time step. Next, we compared the effect of using the constant control fit, constant control MultiStart, and the original MultiStart estimate of the optimal control, see Figure 3.8. Note that our original Model 2 simulation produces the lowest total cost of 101.51, but there is not a drastic difference between the constant fit 107.21 and constant MultiStart of 105.82. In particular, the percentage difference between the total cost of the original Model 2 and constant fit is 5.62% and constant MultiStart is 4.25%.

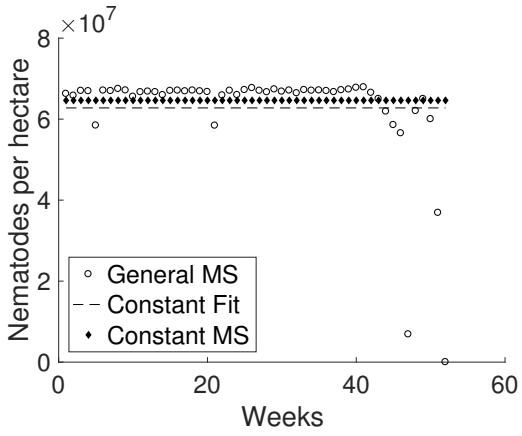


Figure 3.8: A comparison for Model 2 using initial DRW population of 2×10^4 per hectare. We plot the Model 2 MS results with $\mu = \ln(2)$, the linear fit of Model 2 MS, and the constant MS.

4. Discussion

We have modeled the application of pesticide or biological control to regulate an insect pest as a mathematical optimal control problem. In our discrete-time model, the uncontrolled pest population has linear dynamics, and is stage structured. The control acts on the second stage, by reducing survival as a function of applied control effort, and saturates at high density of the control agent because survival cannot be reduced below 0. The cost is quadratic in the pest abundance and linear in the control term. We considered two cases: first, where the control is effective over only one time-step (Model 1), and; second, where the effect of the control decays exponentially over time (Model 2). The solution of the optimal control problem balances the costs of management with the costs of loss of crops, giving an amount of control to apply each time period in order to minimize the total cost. For a specific example, we manage *Diaprepes abbreviatus* using entomopathogenic nematodes as biopesticides, obtaining numerical estimates of an optimal control by using FBS and MultiStart algorithms. FBS is more accurate, but the method breaks down when the control decay is too slow.

There are some facets of our model and approach that can be easily generalized, and some facets that cannot. The formula for the optimal control in terms of the adjoint equations made explicit use of the

underlying linear model, however the specific form of the permitted control function is fairly general as long as it saturates at 1 (all individuals in the targeted stage die). The control needs to act on only one stage-class for the modeling here to work, but which stage-class is targeted for control and the functional form of the control is arbitrary. We chose a quadratic functional form for the cost associated with the pest, both because it heavily penalizes catastrophic loss, and because it is mathematically convenient. The cost of purchasing and applying control is linear in the control variable, which seems a reasonable assumption in the absence of other information.

The numerical algorithm FBS is efficient and accurate for Model 1, but has the disadvantage that it did not produce sensible results for certain parameter values when applied to Model 2. We explained in Section 3.1.3 how and why the method breaks down. As an alternative to FBS we used MultiStart to find the optimal management strategy for Model 2. To evaluate the performance of MultiStart we used this algorithm for Model 1 as well and compared the results with those of FSB. We found that the total cost using MulitStart was was higher compared to FBS, but the difference was small. For instance, for initial DRW population of 2×10^5 per hectare the percentage difference was 4.5% and for initial DRW population of 1.7×10^6 per hectare the percentage difference was 3.4%. Another disadvantage of using MultiStart is that it gives an optimal control which fluctuates more than we would expect (e.g. Figure 3.6). The results get smoother with increasing number of starting values included in MultiStart, but this increases the total running time to find a solution. Alternatively, we used the MultiStart algorithm to obtain the best constant control, and compare this control to the non-constant optimal control.

The cost-effective target pest population size depended on the infestation level at which control was initiated. If control started at low initial pest infestation levels the application of small amounts of control was sufficient to reduce pest populations to very low densities over the entire growing season. In contrast, if control started at relatively high DRW population size the required amount of nematodes to control DRW populations was high and it was not cost effective to reduce infestation to the same low levels as when control started earlier when population size was still small. Our result highlight how beneficial early pest detection is for citrus growers.

We explored how μ affects the total cost. As expected, the longer nematode populations persist in the environment the lower is the cost associated with the control. For instance, if half of the nematode populations survive each time step ($\mu = \ln(2)$), and the initial DRW population is 2×10^5 per hectare, then the total cost of controlling DRW populations is 101. This cost increases by 183% to 287 if nematode populations survive only a single time step (Model 1). This suggests that citrus growers would largely benefit from research on improving nematode population persistence after inundative release.

Using FBS in Model 1 suggests weekly nematode applications that initially increase over time and then decrease. This is a typical result for optimal control, since we are minimizing the total cost for the given

time period, so control declines near the end since the model doesn't have a cost for time greater than T. One way to avoid this would be to expand the time frame over which we optimize (e.g. 10 years instead of one year), but only use the control for the shorter time frame. Using MutiStart the nematode application scheme is relatively constant over most of the growing season (Fig 3.5). The difference in cost is small (e.g. 287 versus 300 for initial DRW population size of 2×10^5 per hectare).

We conclude by highlighting a future avenue of research. Recent research by the present authors [34] has considered theoretical approaches to pest management which appeal to, and develop further, tools from robust control theory. Briefly, robust control seeks to achieve desired dynamic behavior (such as reducing pest population) in the presence of uncertainty or perturbation of the model -facets not addressed here. The downside of robust control tools is that they sacrifice optimality in general. Thus, we plan to compare and contrast the optimal approach adopted here with that of [34], and seek to design methods which incorporate both elements of robust and optimal control.

5. Acknowledgments

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References

[1] D. Atwood, C. Paisley-Jones, Pesticides industry sales and usage 2008–2012 market estimates, US Environmental Protection Agency, Washington, DC Google Scholar.

[2] D. Pimentel, M. Burgess, Environmental and economic costs of the application of pesticides primarily in the united states, in: Integrated pest management, Springer, 2014, pp. 47–71.

[3] L. Ehler, Conservation biological control: past, present, and future, in: Conservation biological control, Elsevier, 1998, pp. 1–8.

[4] J. Pell, J. Hannam, D. Steinkraus, Conservation biological control using fungal entomopathogens, BioControl 55 (1) (2010) 187–198.

[5] T. R. Seastedt, Biological control: Perspectives for maintaining provisioning services in the anthropocene, in: Integrated Pest Management, Springer, 2014, pp. 269–280.

[6] K. E. Grafton-Cardwell, Diaprepes Root Weevil, UCANR Publications, 2004.

[7] D. I. Shapiro-Ilan, D. H. Gouge, A. M. Koppenhöfer, Factors Affecting Commercial Success: Case Studies in Cotton, Turf and Citrus, Entomopathogenic nematology (2002) 333.

[8] R. Bullock, R. Pelosi, E. Killer, Management of citrus root weevils (Coleoptera: Curculionidae) on Florida citrus with soil-applied entomopathogenic nematodes (nematoda: Rhabditida), Florida Entomologist (1999) 1–7.

[9] R. Georgis, A. Koppenhöfer, L. Lacey, G. Bélair, L. Duncan, P. Grewal, M. Samish, L. Tan, P. Torr, R. Van Tol, Successes and failures in the use of parasitic nematodes for pest control, Biological Control 38 (1) (2006) 103–123.

[10] T. E. Miller, B. Tenhumberg, Contributions of demography and dispersal parameters to the spatial spread of a stage-structured insect invasion, Ecological Applications 20 (3) (2010) pp. 620–633.

[11] C. McCoy, R. Stuart, H. Nigg, Seasonal life stage abundance of *diaprepes abbreviatus* in irrigated and non-irrigated citrus plantings in central florida, Florida Entomologist 86 (1) (2003) 34–42.

[12] C. W. McCoy, D. I. Shapiro, L. W. Duncan, K. Nguyen, Entomopathogenic nematodes and other natural enemies as mortality factors for larvae of *diaprepes abbreviatus* (Coleoptera: Curculionidae), Biological Control 19 (2) (2000) 182–190.

[13] K. Jetter, K. Godfrey, Diaprepes root weevil, a new California pest, will raise costs for pest control and trigger quarantines, California Agriculture 63 (3) (2009) 121–126.

[14] J. Macki, A. Strauss, Introduction to optimal control theory, Springer Science & Business Media, 2012.

[15] A. Filippov, On certain questions in the theory of optimal control, Journal of the Society for Industrial and Applied Mathematics, Series A: Control 1 (1) (1962) pp. 76–84.

[16] S. Lenhart, J. Workman, Optimal Control Applied to Biological Models, Taylor & Francis, 2007.

[17] A. Hastings, R. J. Hall, C. M. Taylor, A simple approach to optimal control of invasive species, Theoretical Population Biology 70 (4) (2006) 431–435.

[18] J. Blackwood, A. Hastings, C. Costello, Cost-effective management of invasive species using linear-quadratic control, Ecological Economics 69 (3) (2010) 519–527.

[19] A. Lampert, A. Hastings, E. D. Grosholz, S. L. Jardine, J. N. Sanchirico, Optimal approaches for balancing invasive species eradication and endangered species management, Science 344 (6187) (2014) 1028–1031.

[20] K. Dabbs, Optimal control in discrete pest control models, Master diss., University of Tennessee.
URL http://trace.tennessee.edu/cgi/viewcontent.cgi?article=2367&context=utk_chanhonoproj

[21] A. Whittle, S. Lenhart, K. White, Optimal control of gypsy moth populations, Bulletin of Mathematical Biology 70 (2) (2008) 398–411.

[22] A. J. Whittle, S. Lenhart, L. J. Gross, Optimal control for management of an invasive plant species., Mathematical Biosciences and Engineering: MBE 4 (1) (2007) 101–112.

[23] H. Caswell, Matrix population models, Sinauer Associates, 2001.

[24] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze, E. Mishchenko, The mathematical theory of optimal processes (international series of monographs in pure and applied mathematics), Interscience, New York.

[25] L. J. Mundy, S. P. Jones, D. Crump, J. C. Hervé, A. Konstantinov, F. Utley, D. Potter, S. W. Kennedy, Highly purified hexachlorobenzene induces cytochrome P4501A in primary cultures of chicken embryo hepatocytes, Toxicology and applied pharmacology 248 (3) (2010) 185–193.

[26] R. Campos-Herrera, F. E. El-Borai, L. W. Duncan, It takes a village: entomopathogenic nematode community structure and conservation biological control in Florida (US) orchards, Springer, 2015.

[27] T. Addis, A. Teshome, O. Strauch, R.-U. Ehlers, Life history trait analysis of the entomopathogenic nematode Steinernema feltiae provides the basis for prediction of dauer juvenile yields in monoxenic liquid culture, Applied Microbiology and Biotechnology 100 (10) (2016) pp. 4357–4366.

[28] R. Salguero-Gomez, J. B. Plotkin, Matrix dimensions bias demographic inferences: implications for comparative plant demography, The American Naturalist 176 (6) (2010) 710–722.

[29] Consumer Price Index, 1913-.
URL https://www.minneapolisfed.org/community/teaching-aids/cpi-calculator-information/consumer-price-index-and-inflation-rates-1913

[30] E. Appelbaum, The Consumer Price Index and inflation: adjust numbers for inflation, Journal of Online Mathematics and its Applications.

[31] S. P. Sethi, G. L. Thompson, Optimal control theory: applications to management science and economics, Springer Science & Business Media, 2006.

[32] A. Seierstad, K. Sydsaeter, Optimal control theory with economic applications, Elsevier North-Holland, Inc., 1986.

[33] L. R. de Araujo, D. R. R. Penido, S. C. Júnior, J. L. R. Pereira, P. A. N. Garcia, Comparisons between the three-phase current injection method and the forward/backward sweep method, International Journal of Electrical Power & Energy Systems 32 (7) (2010) pp. 825–833.

[34] C. Guiver, C. Edholm, Y. Jin, M. Mueller, J. Powell, R. Rebarber, B. Tenhumberg, S. Townley, Simple adaptive control for positive linear systems with applications to pest management, SIAM Journal on Applied Mathematics 76 (1) (2016) 238–275.